

AN ALGORITHM TO CONSTRUCT A BASIS FOR THE MODULE OF LOGARITHMIC VECTOR FIELDS

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ABSTRACT. We consider logarithmic vector fields parametrized by finite collections of weighted hyperplanes. For a finite collection of weighted hyperplanes in a two-dimensional vector space, it is known that the set of such vector fields is a free module of rank two whose basis elements are homogeneous. We give an algorithm to construct a homogeneous basis for the module.

1. INTRODUCTION

For an l -dimensional vector space V , a finite collection \mathcal{A} of subspaces of V whose codimensions are one is called a *central hyperplane arrangement* in V . A pair (\mathcal{A}, μ) of a central hyperplane arrangement in V and a map $\mu : \mathcal{A} \rightarrow \mathbb{Z}_{>0}$ is called a *multiarrangement* in V . Let V^* be the dual space of V , and S the algebra of polynomial functions on V . The algebra S has a natural graded structure. For a multiarrangement (\mathcal{A}, μ) in V , we define $D(\mathcal{A}, \mu)$ to be the set of derivations θ of S satisfying the following condition: $\theta(\alpha)$ is in the ideal generated by $\alpha^{\mu(\ker(\alpha))}$ for each $\alpha \in S$ with $\ker(\alpha) \in \mathcal{A}$. The set $D(\mathcal{A}, \mu)$ also has a natural structure of a graded S -module. We say that a multiarrangement (\mathcal{A}, μ) is free if the corresponding module $D(\mathcal{A}, \mu)$ is free. For a free multiarrangement (\mathcal{A}, μ) , $D(\mathcal{A}, \mu)$ has a homogeneous basis. The multi-set of degrees of a homogeneous basis for $D(\mathcal{A}, \mu)$ is called the exponents of (\mathcal{A}, μ) .

Ziegler showed that $D(\mathcal{A}, \mu)$ is free for a multiarrangement (\mathcal{A}, μ) in a two-dimensional vector space [8]. Yoshinaga proved a theorem which characterizes the free multiarrangements in a three-dimensional vector space. In the theorem, exponents of multiarrangements in a two-dimensional vector space play an important role [7]. For some multiarrangements (\mathcal{A}, μ) in a two-dimensional vector space, homogeneous bases for $D(\mathcal{A}, \mu)$ are explicitly given. Hence exponents for these multiarrangements are also explicitly given. (See [5, 6] and so on.)

The set of multiarrangements has a structure of a graded poset. (See Section 2.) The mapping (\mathcal{A}, μ) to $D(\mathcal{A}, \mu)$ is order-reversing. In this paper, for multiarrangements (\mathcal{A}, μ) and (\mathcal{A}, μ') in a two-dimensional vector space such that (\mathcal{A}, μ') is larger than (\mathcal{A}, μ) , we show an algorithm to construct a homogeneous basis for $D(\mathcal{A}, \mu')$ from a homogeneous basis for $D(\mathcal{A}, \mu)$.

In Section 2, we recall the definition of the module of logarithmic vector fields and K. Saito's criterion. In Section 3, we show algorithms to construct a homogeneous basis for $D(\mathcal{A}, \mu)$ for a multiarrangement (\mathcal{A}, μ) in a two-dimensional vector space. In Section 4, we show some applications of our algorithms. In Subsection 4.1, we consider rise and fall of the difference of exponents. In Subsection 4.2, we consider the case where the base field is finite, and explicitly describe a homogeneous basis for some $D(\mathcal{A}, \mu)$. In Subsection 4.3, an implementation of our algorithms as a program of the computer algebra system `risa/asir` is outlined.

2. DEFINITION AND NOTATION

Let \mathbb{K} be a field, V a two-dimensional vector space over \mathbb{K} , and S a polynomial ring $\mathbb{K}[x, y]$. The algebra S naturally has a structure $S = \bigoplus_{i \in \mathbb{N}} S_i$ of a graded algebra, where S_i is a vector space whose basis is $\{x^j y^{i-j} \mid j = 0, \dots, i\}$.

We call a finite collection \mathcal{A} of hyperplanes containing the origin in V a *central arrangement*. A pair (\mathcal{A}, μ) of a central arrangement \mathcal{A} and a map μ from \mathcal{A} to the set of positive integers $\mathbb{Z}_{>0}$ is called a *multiarrangement*. We identify a multiarrangement with a map μ from the set of hyperplanes in V containing the origin to the set of nonnegative integers \mathbb{N} whose support $\text{supp}(\mu) = \{H \mid \mu(H) \neq 0\}$ is a finite set. We write $|\mu|$ for $\sum_H \mu(H)$. We define $(\mathcal{A}, \mu) \subset (\mathcal{A}', \mu')$ if μ and μ' satisfy $\mu(H) \leq \mu'(H)$ for all $H \in \mathcal{A}$. The set of multiarrangements has a structure of a graded poset with the minimum element $(\emptyset, 0)$ by the relation \subset , where 0 is the map such that $0(H) = 0$ for all H .

Definition 2.1. For a multiarrangement (\mathcal{A}, μ) , we define the set $D(\mathcal{A}, \mu)$ of logarithmic vector fields to be

$$\left\{ \theta = f(x, y)\partial_x + g(x, y)\partial_y \left| \begin{array}{l} f(x, y), g(x, y) \in S. \\ \theta(\alpha) \text{ is in the ideal } (\alpha^{\mu(\ker(\alpha))}) \\ \text{for each } \alpha \in S_1 \setminus \{0\}. \end{array} \right. \right\},$$

where ∂_x and ∂_y respectively denote the partial differential operators in the variables x and y .

For a multiarrangement (\mathcal{A}, μ) , $D(\mathcal{A}, \mu)$ has a structure $D(\mathcal{A}, \mu) = \bigoplus_{i \in \mathbb{N}} D(\mathcal{A}, \mu)_i$ of a graded S -module, where

$$D(\mathcal{A}, \mu)_i = \{ f(x, y)\partial_x + g(x, y)\partial_y \in D(\mathcal{A}, \mu) \mid f(x, y), g(x, y) \in S_i \}.$$

The mapping $(\mathcal{A}, \mu) \mapsto D(\mathcal{A}, \mu)$ is order-reversing. Namely, by definition, $D(\mathcal{A}, \mu) \subset D(\mathcal{A}', \mu')$ for multiarrangements $(\mathcal{A}', \mu') \subset (\mathcal{A}, \mu)$.

We consider only the case of two-dimensional vector spaces in this paper. In this case, it is known that $D(\mathcal{A}, \mu)$ is always a free S -module whose rank is two, and that $D(\mathcal{A}, \mu)$ has a homogeneous basis. We give an algorithm to construct a basis for $D(\mathcal{A}, \mu)$ in this paper. The following is a well-known criterion.

Theorem 2.2 (K. Saito's criterion [4]). *Let $\{\theta_i \in D(\mathcal{A}, \mu)_{d_i} \mid i = 1, 2\}$ be S -linearly independent. Then $\{\theta_1, \theta_2\}$ is a basis for $D(\mathcal{A}, \mu)$ if and only if $|\mu| = \sum_i d_i$.*

3. MAIN RESULT

In this section, we give algorithms to construct a homogeneous basis for $D(\mathcal{A}, \mu)$.

First we consider the case where $\mu \subset \mu'$ and $|\mu'| = |\mu| + 1$. We show an algorithm to construct a homogeneous basis for $D(\mathcal{A}', \mu')$ from a homogeneous basis for $D(\mathcal{A}, \mu)$.

Theorem 3.1. *Let (\mathcal{A}, μ) be a multiarrangement. Fix $\alpha = \alpha_x x + \alpha_y y \in S_1 \setminus \{0\}$. Let \mathcal{A}' be $\mathcal{A} \cup \ker(\alpha)$, and μ' the map such that*

$$\mu'(H) = \begin{cases} \mu(\ker(\alpha)) + 1 & (H = \ker(\alpha)) \\ \mu(H) & (H \in \mathcal{A} \setminus \{\ker(\alpha)\}) \end{cases}.$$

Then we can construct a homogeneous basis (θ'_1, θ'_2) for $D(\mathcal{A}', \mu')$ from a homogeneous basis (θ_1, θ_2) for $D(\mathcal{A}, \mu)$ by the following algorithm ALG1 :

Input: $\theta_1, \theta_2, \alpha = (\alpha_x x + \alpha_y y), m = \mu(\ker(\alpha))$.

Output: θ'_1, θ'_2 .

Procedure:

- (1) If $\deg(\theta_1) < \deg(\theta_2)$, then swap θ_1 and θ_2 .
- (2) Let $g(x, y) = \frac{\theta_2(\alpha)}{\alpha^m}$.
- (3) If $g(\alpha_y, -\alpha_x) = 0$, then let $\theta'_1 = \alpha \cdot \theta_1$ and $\theta'_2 = \theta_2$, and finish the procedure.
- (4) Let $f(x, y) = \frac{\theta_1(\alpha)}{\alpha^m}$.
- (5) If $f(\alpha_y, -\alpha_x) = 0$, then let $\theta'_1 = \theta_1$ and $\theta'_2 = \alpha \cdot \theta_2$, and finish the procedure.
- (6) Let $q(x, y)$ be a homogeneous polynomial in variables x, y of degree $\deg(\theta_1) - \deg(\theta_2)$ satisfying the equation

$$f(\alpha_y, -\alpha_x) + g(\alpha_y, -\alpha_x)q(\alpha_y, -\alpha_x) = 0.$$

- (7) Let $\theta'_1 = \theta_1 + q(x, y) \cdot \theta_2$ and $\theta'_2 = \alpha \cdot \theta_2$.

Proof. Since $\theta_1, \theta_2 \in D(\mathcal{A}, \mu)$, both $f(x, y) = \frac{\theta_1(\alpha)}{\alpha^m}$ and $g(x, y) = \frac{\theta_2(\alpha)}{\alpha^m}$ are polynomials.

First we consider the case $g(\alpha_y, -\alpha_x) = 0$. Since θ_2 is homogeneous, $g(\alpha_y, -\alpha_x)$ is also homogeneous. Since \mathbb{K} is a field, $g(\alpha_y, -\alpha_x) = 0$ if and only if $g(x, y)$ is divisible by α . This implies $\theta_2(\alpha) \in \alpha^{m+1}S$. Hence θ_2 is in $D(\mathcal{A}', \mu')$. Since $\theta_1 \in D(\mathcal{A}, m)$, $\alpha \cdot \theta_1$ is also in $D(\mathcal{A}', \mu')$. Since θ_1 and θ_2 are linearly independent, $\alpha \cdot \theta_1$ and θ_2 are linearly independent. It is clear that $\deg(\alpha \cdot \theta_1) + \deg(\theta_2) = \deg(\theta_1) + \deg(\theta_2) + 1$. Hence $(\alpha \cdot \theta_1, \theta_2)$ is a basis for $D(\mathcal{A}', \mu')$ by Theorem 2.2.

Next we consider the case $g(\alpha_y, -\alpha_x) \neq 0$. Let $d = \deg(\theta_1) - \deg(\theta_2)$. It follows from $g(\alpha_y, -\alpha_x) \neq 0$ that the equation

$$f(\alpha_y, -\alpha_x) + g(\alpha_y, -\alpha_x) \sum_{i=0}^d q_i \alpha_y^i (-\alpha_x)^{d-i} = 0$$

is solvable. For example,

$$(1) \quad \begin{cases} q_0 = -\frac{f(\alpha_y, -\alpha_x)}{g(\alpha_y, -\alpha_x)(-\alpha_x)^d} - \sum_{i=1}^d \frac{\alpha_y^i}{(-\alpha_x)^i} & (i = 0) \\ q_i = 1 & (i > 0) \end{cases}$$

is one of solutions if $\alpha_x \neq 0$. For such q_i , let $q(x, y) = \sum_{i=0}^d q_i x^i y^{d-i}$ and $\theta'_2 = \theta_1 + q(x, y) \cdot \theta_2$. Since $\theta'_1(\alpha) = \theta_1(\alpha) + q(x, y) \cdot \theta_2(\alpha)$,

$$\begin{aligned} \frac{\theta'_1(\alpha)}{\alpha^m} &= \frac{\theta_1(\alpha)}{\alpha^m} + \frac{q(x, y) \cdot \theta_2(\alpha)}{\alpha^m} \\ &= f(x, y) + q(x, y)g(x, y). \end{aligned}$$

Since $\frac{\theta'_1(\alpha)}{\alpha^m}$ is homogeneous, and $f(\alpha_y, -\alpha_x) + q(\alpha_y, -\alpha_x)g(\alpha_y, -\alpha_x) = 0$, $\frac{\theta'_1(\alpha)}{\alpha^m}$ is divisible by α . Hence we have $\theta'_1 \in D(\mathcal{A}', \mu')$. Since $\theta_2 \in D(\mathcal{A}, \mu)$, $\alpha \cdot \theta_2 \in D(\mathcal{A}', \mu')$. The linearly independence of $\{\theta_1, \theta_2\}$ implies the linearly independence of $\{\theta'_2 = \theta_1 + q(x, y) \cdot \theta_2, \alpha \cdot \theta_2\}$. It is clear that $\deg(\theta'_1) + \deg(\alpha \cdot \theta_2) = \deg(\theta_1) + \deg(\theta_2) + 1$. Hence $(\theta'_1, \alpha \cdot \theta_2)$ is a basis for $D(\mathcal{A}', \mu')$. \square

By taking the polynomial defined by (1) as $q(x, y)$ in Step (6) of the algorithm **ALG1**, we have the following algorithm.

Corollary 3.2. *Let (\mathcal{A}, μ) be a multiarrangement. Fix $\alpha = \alpha_x x + \alpha_y y \in S_1 \setminus \{0\}$. Let \mathcal{A}' be $\mathcal{A} \cup \ker(\alpha)$, and μ' the map such that*

$$\mu'(H) = \begin{cases} \mu(\ker(\alpha)) + 1 & (H = \ker(\alpha)) \\ \mu(H) & (H \in \mathcal{A} \setminus \{\ker(\alpha)\}). \end{cases}$$

*Then we can construct a homogeneous basis (θ'_1, θ'_2) for $D(\mathcal{A}', \mu')$ from a homogeneous basis (θ_1, θ_2) for $D(\mathcal{A}, \mu)$ by the following algorithm **ALG2** :*

Input: $\theta_1, \theta_2, \alpha = (\alpha_x x + \alpha_y y), m = \mu(\ker(\alpha))$.

Output: θ'_1, θ'_2 .

Procedure:

- (1) If $\deg(\theta_1) < \deg(\theta_2)$, then swap θ_1 and θ_2 .
- (2) Let $g(x, y) = \frac{\theta_2(\alpha)}{\alpha^m}$.
- (3) If $g(\alpha_y, -\alpha_x) = 0$, then let $\theta'_1 = \alpha \cdot \theta_1$ and $\theta'_2 = \theta_2$, and finish the procedure.
- (4) Let $f(x, y) = \frac{\theta_1(\alpha)}{\alpha^m}$.
- (5) If $f(\alpha_y, -\alpha_x) = 0$, then let $\theta'_1 = \theta_1$ and $\theta'_2 = \alpha \cdot \theta_2$, and finish the procedure.
- (6) Let $d = \deg(\theta_1) - \deg(\theta_2)$.

(7) If $\alpha_x = 0$, then let $\theta'_1 = \theta_1 - \frac{f(1,0)}{g(1,0)}x^d\theta_2$ and $\theta'_2 = y \cdot \theta_2$, and finish the procedure.

(8) Let $q(x, y)$ be

$$\left(-\frac{f(\alpha_y, -\alpha_x)}{g(\alpha_y, -\alpha_x)(-\alpha_x)^d} - \sum_{i=1}^d \frac{\alpha_y^i}{(-\alpha_x)^i} \right) y^d + \sum_{i=1}^d x^i y^{d-i}.$$

(9) Let $\theta'_1 = \theta_1 + q(x, y) \cdot \theta_2$ and $\theta'_2 = \alpha \cdot \theta_2$.

We have a basis (∂_x, ∂_y) for $D(\emptyset, 0)$. By applying the algorithm in Theorem 3.1 recursively, we can construct a basis for $D(\mathcal{A}, \mu)$ for all multiarrangements.

Theorem 3.3. *We can construct a homogeneous basis (θ_1, θ_2) for a multiarrangement $D(\mathcal{A}, \mu)$ by the following algorithm ALG3 :*

Input: (\mathcal{A}, μ) .

Output: (θ_1, θ_2) .

Procedure:

- (1) If $\mathcal{A} = \emptyset$, then let $\theta_1 = \partial_x$ and $\theta_2 = \partial_y$, and finish the procedure.
- (2) Let H be a hyperplane in $\text{supp}(\mu)$.
- (3) Let μ' be a map such that

$$\mu'(H') = \begin{cases} \mu(H) - 1 & (H' = H), \\ \mu(H') & (H' \in \mathcal{A} \setminus \{H\}). \end{cases}$$

- (4) Let $\mathcal{A}' = \text{supp}(\mu')$.
- (5) Let (θ'_1, θ'_2) be the resulting basis of ALG3(\mathcal{A}', μ').
- (6) Let $\alpha \in S_1$ be a linear form such that $\ker(\alpha) = H$.
- (7) Let (θ_1, θ_2) be the resulting basis of ALG1($\theta'_1, \theta'_2, \alpha, \mu'(H)$).

4. APPLICATION

In this section, we show applications of our algorithms. First we consider the difference between the degrees of elements of homogeneous basis. Next we explicitly describe a homogeneous basis for some arrangements in the case where \mathbb{K} is a finite field. Finally we briefly introduce an implementation of our algorithms.

4.1. Difference between exponents. In this subsection, we consider the difference between the degrees of elements of homogeneous basis, i.e., the difference between the exponents of a multiarrangement in two-dimensional vector space. When a multiarrangement is made larger, the difference of its exponents either increases or decreases. We show that the difference decreases if a generic hyperplane is added to a multiarrangement with the two degrees different.

The next two corollaries follow from Theorem 3.1.

Corollary 4.1. *Let (\mathcal{A}, μ) be a multiarrangement, and $\{\theta_1, \theta_2\}$ a homogeneous basis for $D(\mathcal{A}, \mu)$ such that $\deg(\theta_1) \geq \deg(\theta_2)$.*

Fix $\alpha = \alpha_x x + \alpha_y y \in S_1 \setminus \{0\}$. Let μ' be the map such that

$$\mu'(H) = \begin{cases} \mu(\ker(\alpha)) + 1 & (H = \ker(\alpha)) \\ \mu(H) & (H \in \mathcal{A} \setminus \{\ker(\alpha)\}). \end{cases}$$

Let $\{\theta'_1, \theta'_2\}$ be a homogeneous basis for $D(\text{supp } \mu', \mu')$.

Let $d' = |\deg(\theta'_1) - \deg(\theta'_2)|$, $d = |\deg(\theta_1) - \deg(\theta_2)|$. Then

$$\begin{cases} d' > d & (g(\alpha_y, -\alpha_x) = 0 \text{ or } d = 0) \\ d' < d & (\text{otherwise}), \end{cases}$$

where $g = \frac{\theta_2(\alpha)}{\alpha^{\mu(\ker(\alpha))}}$.

Proof. The corollary directly follows from the algorithm **ALG1** in Theorem 3.1. \square

Corollary 4.2. *Let (\mathcal{A}, μ) be a multiarrangement, and $\{\theta_1, \theta_2\}$ a homogeneous basis for $D(\mathcal{A}, \mu)$ such that $\deg(\theta_1) \geq \deg(\theta_2)$. Let $\theta_2 = \varphi(x, y)\partial_x + \psi(x, y)\partial_y$. For α_x, α_y satisfying*

$$\alpha_x \varphi(\alpha_y, -\alpha_x) + \alpha_y \psi(\alpha_y, -\alpha_x) \neq 0,$$

let $\alpha = \alpha_x x + \alpha_y y$, $\mathcal{A}' = \mathcal{A} \cup \{\ker(\alpha)\}$, and (\mathcal{A}', μ') a multiarrangement such that

$$\mu'(H) = \begin{cases} 1 & (H = \ker(\alpha)) \\ \mu(H) & (H \in \mathcal{A} \setminus \{\ker(\alpha)\}). \end{cases}$$

If $\deg(\theta_1) - \deg(\theta_2) = 1$, then $\deg(\theta'_1) = \deg(\theta'_2)$, where $\{\theta'_1, \theta'_2\}$ is a homogeneous basis for $D(\mathcal{A}', \mu')$.

Proof. Since

$$\theta_2(\alpha_x x + \alpha_y y)|_{x=\alpha_y, y=-\alpha_x} = \alpha_x \varphi(\alpha_y, -\alpha_x) + \alpha_y \psi(\alpha_y, -\alpha_x) \neq 0,$$

$\theta_2(\alpha)$ does not divide by $\alpha = \alpha_x x + \alpha_y y$. Since $\theta_2 \in D(\mathcal{A}, \mu)$, $\mu(\ker(\alpha)) = 0$. Apply **ALG1**. Since

$$g(\alpha_x, -\alpha_y) = \frac{\theta_2(\alpha_x x + \alpha_y y)}{\alpha^0} \Big|_{x=\alpha_y, y=-\alpha_x} \neq 0,$$

$$|\deg(\theta'_1) - \deg(\theta'_2)| < |\deg(\theta_1) - \deg(\theta_2)|$$

if $\deg(\theta_1) > \deg(\theta_2)$, where $\{\theta'_1, \theta'_2\}$ is the homogeneous basis for $D(\mathcal{A}', \mu')$ obtained from $(\theta_1, \theta_2, \alpha, 0)$ by the algorithm **ALG1**. Hence we have the corollary. \square

Corollary 4.1 implies the following corollary.

Corollary 4.3. *Fix $H = \ker(\alpha) \in \mathcal{A}$. Let μ satisfy*

$$2\mu(H) > |\mu|.$$

If $\{\theta_1, \theta_2\}$ is a basis for $D(\mathcal{A}, \mu)$ satisfying $\deg(\theta_1) > \deg(\theta_2)$, then $\{\alpha^n \theta_1, \theta_2\}$ is a basis for $D(\mathcal{A}, \mu'')$, where μ'' is the map such that

$$\mu''(H') = \begin{cases} \mu(H) + n & (H' = H) \\ \mu(H') & (H' \in \mathcal{A} \setminus \{H\}). \end{cases}$$

Proof. It is enough to show the case where $n = 1$. In this case, $2\mu'(H) > |\mu'|$, $2\mu(H) > |\mu|$, and $\mu'(H) = \mu(H) + 1$. For a multiarrangement (\mathcal{A}, κ) such that $2\kappa(H) \geq |\kappa|$ for some $H \in \mathcal{A}$, it is known that the exponents of (\mathcal{A}, κ) are $(\kappa(H), |\kappa| - \kappa(H))$. (See [3].) Since $2\mu(H) > |\mu|$ and $2\mu'(H) > |\mu'|$, we have

$$\begin{aligned} \deg(\theta_1) &= \mu(H), \\ \deg(\theta_2) &= |\mu| - \mu(H), \\ \deg(\theta'_1) &= \mu'(H) = \mu(H) + 1, \\ \deg(\theta'_2) &= |\mu'| - \mu'(H) = |\mu| - \mu(H). \end{aligned}$$

Hence $\deg(\theta'_1) - \deg(\theta'_2) = 2\mu(H) - |\mu| + 1 = \deg(\theta_1) - \deg(\theta_2) + 1$. Since the difference increases, $g(\alpha_y, -\alpha_x) = 0$ by Corollary 4.1. Since $g(\alpha_y, -\alpha_x) = 0$, $\{\alpha \theta_1, \theta_2\}$ is a basis for $D(\mathcal{A}, \mu'')$ by Theorem 3.1. Hence we have the corollary. \square

4.2. Finite Fields. In this subsection, let p be a prime number. Let us take the finite field F_q consisting of $q = p^n$ elements as \mathbb{K} . Let \mathcal{A} be the set of hyperplanes in F_q^2 . We explicitly describe homogeneous bases for some multiarrangements $D(\mathcal{A}, \mu)$.

Let θ_{q^i} be $x^{q^i} \partial_x + y^{q^i} \partial_y$, and κ_{q^i} the map such that $\kappa_{q^i}(H) = q^i$ for all hyperplanes H .

Lemma 4.4. *The set $\{\theta_{q^i}, \theta_{q^{i+1}}\}$ is a basis for $D(\mathcal{A}, \kappa_{q^i})$.*

Proof. Since $(ax + by)^q = ax^q + by^q$, θ_{q^i} and $\theta_{q^{i+1}}$ are in $D(\mathcal{A}, \kappa_{q^i})$. Since $x^{q^i} y^{q^{i+1}} - x^{q^{i+1}} y^{q^i} \neq 0$, $\{\theta_{q^i}, \theta_{q^{i+1}}\}$ is linearly independent. Since the number $|\mathcal{A}|$ of hyperplanes in F_q^2 is $\frac{|F_q^2 \setminus (0,0)|}{F_q^\times} = \frac{q^2 - 1}{q - 1} = q + 1$, $|\kappa_{q^i}| = q^i(q + 1) = \deg(\theta_{q^i}) + \deg(\theta_{q^{i+1}})$. Hence the set $\{\theta_{q^i}, \theta_{q^{i+1}}\}$ is a basis for $D(\mathcal{A}, \kappa_{q^i})$. \square

Corollary 4.5. *Let i be an positive integer. For each $H \in \mathcal{A}$, fix $\alpha_H(x, y) \in S_1$ satisfying $\ker(\alpha_H(x, y)) = H$, and let j_H be an integer such that $0 \leq j_H \leq q^{i+1} - q^i$. Let $\alpha = \prod_H \alpha_H^{j_H}$. Let μ be the map from \mathcal{A} to $\mathbb{Z}_{>0}$ such that $\mu(H) = q^i + j_H$ for each hyperplane H .*

Then

$$\{\alpha(x, y) \cdot \theta_{q^i}, \theta_{q^{i+1}}\}$$

is a basis for $D(\mathcal{A}, \mu)$.

Proof. Fix a saturated chain

$$((\mathcal{A}, \mu_0) = (\mathcal{A}, \kappa_{q^i}), (\mathcal{A}, \mu_1), \dots, (\mathcal{A}, \mu_n) = (\mathcal{A}, \mu))$$

of multiarrangements. For $0 \leq k \leq n$, let $\alpha_{\mu_k} = \prod_{H \in \mathcal{A}} \alpha_H^{\mu_k(H) - \mu_0(H)}$, and $\theta'_k = \alpha_{\mu_k} \theta_{q^i}$. For $0 \leq k < n$, let $\alpha_k = \frac{\alpha_{\mu_{k+1}}}{\alpha_{\mu_k}}$, and $m_k = \mu_k(\ker(\alpha_k))$. Since $\theta_{q^{i+1}}$ is in $D(\mathcal{A}, \mu_{k+1})$, the resulting basis for $D(\mathcal{A}, \mu_{k+1})$ of the algorithm **ALG2**($\theta'_k, \theta_{q^{i+1}}, \alpha_k, m_k$) is $\{\theta'_{k+1}, \theta_{q^{i+1}}\}$ if $\{\theta'_k, \theta_{q^{i+1}}\}$ is a basis for $D(\mathcal{A}, \mu_k)$. Since the set $\{\theta_{q^i}, \theta_{q^{i+1}}\}$ is a basis for $D(\mathcal{A}, \kappa_{q^i}) = D(\mathcal{A}, \mu_0)$, we have the corollary. \square

4.3. Implementation. A simple implementation of our algorithms as a program for the computer algebra system **risa/asir**[1] is available in [2]. The implementation is useful to compute exponents for many examples. For example, the following can be observed with the implementation.

Proposition 4.6. *Let $H_1 = \{x + y = 0\}$, $H_2 = \{x - y = 0\}$, $H_3 = \{x = 0\}$, $H_4 = \{y = 0\}$, and $\mathcal{A} = \{H_i\}$. Let us assume that*

$$\mu(H_i) < \frac{|\mu|}{2}$$

for all i . Then, for $20 \leq \mu(H_i) \leq 30$, $d = 2$ if and only if there exist $k, h, l \in \mathbb{Z}$ such that

$$\begin{aligned} &(\mu(H_1) = 2k + 3 + 4h, \quad \mu(H_2) = 2k + 1, \quad \mu(H_3) = \mu(H_4) = 2l), \\ &(\mu(H_3) = 2k + 3 + 4h, \quad \mu(H_4) = 2k + 1, \quad \mu(H_1) = \mu(H_2) = 2l), \\ &(\mu(H_1) = 2k + 1 + 4h, \quad \mu(H_2) = 2k + 1, \quad \mu(H_3) = \mu(H_4) = 2l + 1), \end{aligned}$$

or

$$(\mu(H_3) = 2k + 1 + 4h, \quad \mu(H_4) = 2k + 1, \quad \mu(H_1) = \mu(H_2) = 2l + 1).$$

Remark 4.7. For this computation, we used **risa/asir** version 20050209 (Kobe Distribution) on Linux machine (CPU: Intel(R) Celeron(R) CPU 2.26GHz, Memory: 494M, bogomips: 4521.98). The 14641 examples was computed in 40 minutes.

Acknowledgments. The author would like to thank Professor Hiroaki Terao for suggesting Lemma 4.4 and Corollary 4.5 for special cases.

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